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Local Extrema of Traces of Heat Kernels on S^2

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On S^2 we consider metrics conformal to the standard round metric g and of area 4π. We show that among such metrics the trace of the heat kernel $Tr(e^{t\Delta})$ is locally minimized at g, for any given $t > 0$. The local condition is expressed in terms of an L^∞ neighborhood of the set of conformal factors of g of the form $|\tau'|$, with τ a Möbius transformation. To prove this result we use a power series expansion of the trace in terms of its conformal variations. We derive a combinatorial formula for

1. INTRODUCTION AND MAIN RESULTS

The standard round metric on S^2 with constant curvature 1 will be denoted by g. A metric is *conformal* to g if it has the form $g_w = wg$; by the uniformization theorem every other smooth metric can be pulled back via a diffeomorphism to some such metric. Denote by Δ and Δ_w the Laplace–Beltrami operators corresponding to the metrics g and g_w; it is well known that $\Delta_w = w^{-1}\Delta$. The sequence of eigenvalues of Δ_w will be denoted by $0 = \lambda_0(w) < \lambda_1(w) \leqslant \lambda_2(w) \leqslant \cdots \uparrow \infty$. The L^2 trace of the heat semigroup $e^{t\Delta_w}$ is then given by

$$Tr(e^{t\Delta_w}) = \int_{S^2} p_t^w(x, x)\, w(x)\, dx = \sum_{j=0}^{\infty} e^{-\lambda_j(w)\, t},$$

where p_t^w is the smooth kernel of $e^{t\Delta_w}$, dx is the area element in the metric g and $w\, dx$ the one in the metric wg.

The purpose of this paper is to study the following extremal problem: *Is it true that if $t > 0$ and $\int_{S^2} w\, dx = 4\pi$ then*

$$Tr(e^{t\Delta_w}) \geqslant Tr(e^{t\Delta}), \tag{1}$$

with equality only when (S^2, g_w) is isometric to (S^2, g)?

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The first evidence in favor of (1), at least for small values of t, is provided by the asymptotic expansion of the trace

$$Tr(e^{t\Delta_w}) = \frac{1}{4\pi t}\int_{S^2} w\,dx + \frac{1}{3} + \frac{t}{60\pi}\int_{S^2} K_w^2 w\,dx + O(t^2)$$

where K_w is the Gaussian curvature of (M, g_w) (see for example [B-G-M], III.E.IV). Thus, under the are constraint $\int_{s^2} w = 4\pi$ a comparison with the corresponding expansion for $Tr(e^{t\Delta})$ gives

$$Tr(e^{t\Delta_w}) - Tr(e^{t\Delta}) = \frac{t}{60\pi}\left(\int_{S^2} K_w^2 w\,dx - 4\pi\right) + O(t^2).$$

A simple application of the Cauchy-Schwartz inequality coupled with the Gaussian–Bonnet theorem yields $\int_{S^2} K_w^2 w\,dx \geqslant 4\pi$, with equality if and only if K_w is constant. Thus, for any given w, the trace inequality (1) holds at least for t sufficiently small, say $0 < t \leqslant t_w$, with no obvious lower bounds on t_w (for example independent of w).

A second piece of evidence is provided by the eigenvalue expression of the trace, which shows that the validity of (1) for large t is equivalent to a comparison theorem for the first eigenvalue $\lambda_1(w) \leqslant \lambda_1(1) = 2$, whenever $\int_{S_2} w = 4\pi$ and with equality for $g_w = g$, up to isometries. This has been proved by J. Hersch [H] and extended later by P. Li and S. T. Yau to the real projective space $\mathbf{R}P^2$ and to the square torus T^2 ([L-Y], p. 283).

Our main result below states that for each $t > 0$ inequality (1) holds *locally*, that is for all metrics g_w such that (S^2, g_w) is not too far from being isometric to the standard sphere. In order to make this statement precise let us recall that the set of metrics on S^2 which are conformal to g decomposes into isometry classes under the action of the Möbius group, given by $w \to w_\tau$ with

$$w_\tau := (w \circ \tau)\,|\tau'|,$$

where τ is Möbius and $|\tau'|$ denotes the determinant of the Jacobian of τ. In particular, (S^2, g_w) is isometric to the standard (S^2, g) if and only if $w = |\tau'|$, some τ.

We define the amount by which (S^2, g_w) differs from being isometric to the standard round sphere (S^2, g) by

$$M[w] = \inf\{\,\|w_\tau - 1\|_\infty : \tau \text{ Möbius}\}.$$

The set $\{w \in C^\infty : w > 0, M[w] < \varepsilon\}$ is open in the L^∞ norm, and contains all the densities $|\tau'|$. Observe also that $M[w] = \inf_\tau \|w\,|\tau'|^{-1} - 1\|_\infty$ and that $M[w] = 0$ if and only if $w = |\tau'|$, some τ.

Our local minimum theorem for the trace is the following.

THEOREM 1. *In the class of metrics on* S^2 *of the form* $g_w = wg$ *and with given area,* $Tr(e^{t\Delta_w})$ *has a local minimum at* g. *More precisely, given* $b \geqslant a > 0$ *there exists an* $\varepsilon_0 = \varepsilon_0(a, b)$ *such that for* $0 < w \in C^\infty(S^2)$ *with* $\int_{S^2} w = 4\pi$ *and* $M[w] < \varepsilon_0$ *we have*

$$Tr(e^{t\Delta_w}) \geqslant Tr(e^{t\Delta}), \qquad \forall t \in [a, b]$$

with equality for some t *only when* $M[w] = 0$, *i.e., at the standard metric, up to isometries.*

Remark 1. We observe that Theorem 1 extends immediately to the real projective space. In this case $M[w] = \|w - 1\|_\infty$, since the only conformal diffeomorphisms are the rotations.

We were originally led to conjecture the trace inequality (1) by results of Onofri [O] and Osgood–Phillips–Sarnak [O-P-S] about determinants of Laplacians. In a general compact surface the determinant is defined in terms of the zeta function $Z(s) = \sum_1^\infty \lambda_j^{-s}$, where the λ_j's denote the eigenvalues of Δ; through a regularization process one can continue $Z(s)$ to a meromorphic function in the plane, regular at $s = 0$. The determinant is defined as $\det \Delta = e^{-Z'(0)}$, formally the product of the nonzero eigenvalues. In [O] Onofri proved that among all metrics on S^2 conformal to the standard metric g, the determinant is minimized precisely at g; this result was later generalized by Osgood–Phillips–Sarnak [O-P-S] to general compact surfaces. To make contact with our conjecture (1) we observe that by a simple Mellin transform argument the extremal result for $\det \Delta$ can be recast as the following integral inequality

$$\int_0^\infty [Tr(e^{t\Delta_w}) - Tr(e^{t\Delta})] \frac{dt}{t} \geqslant 0$$

whenever $w > 0$, $\int_{S^2} w = 4\pi$ and with equality if and only if $w = |\tau'|$, some Möbius transformation τ. What enabled Onofri to prove his result was the fact that the quantity on the left side can be explicitly computed in terms of w ([O], [O-P-S]) via the Polyakov formula, reducing the problem to the determination of the best constant in the Moser-Trudinger inequality.

Recently [M3], the author derived another related inequality:

$$\int_0^\infty [Tr(e^{t\Delta_w}) - Tr(e^{t\Delta})]\, dt \geqslant 0$$

valid, again, for all smooth $w > 0$ with mean 4π, and with equality if and only if wg is isometric to g. It was possible to compute the LHS explicitly in terms of w and the resulting functional was precisely the one entering in the sharp logarithmic Hardy–Littlewood–Sobolev inequality, derived by Beckner [Bec] and Carlen–Loss [C-L].

Thus, "integrated" versions of (1) are certainly true globally, and the ways they are solved show the nontrivially of our trace conjecture: its solution must somehow encapsulate the process leading to sharp Moser–Trudinger and logarithmic Sobolev inequalities on S^2.

To prove Theorem 1 we shall use perturbation results derived in [M1]. The main result established there is that $Tr(e^{t\Delta_w})$ is an analytic function of w, the conformal factor. What we mean by this is that given any $w_0 > 0$ the trace has a Taylor expansion $Tr(e^{t\Delta_w}) = \sum_0^\infty 1/k!\; I_t^k(w - w_0)$, where I_t^k is the kth conformal differential, or variation, of the trace at w_0, i.e. $I_t^k[\phi] := (d^k/d\varepsilon^k)|_{\varepsilon=0}\, Tr(e^{t\Delta_{w_0+\varepsilon\phi}})$. The k-linear forms I_t^k were explicitly computed and estimated to show that the perturbation series is actually convergent for $\|w - w_0\|_\infty$ small enough. In particular, for $0 < t \leqslant T$ we have the following second order expansion ([M1], Theorem 2)

$$Tr(e^{-\Delta_{1-\phi}}) = Tr(e^{t\Delta}) + t\frac{d}{dt}\int_{S^2} p_t(x,x)\,\phi(x)\,dx$$
$$+\frac{t}{2}\frac{d^2}{dt^2}(H_t\phi,\phi)_{L^2(dx)} + t^{-1}O(\|\phi\|_\infty\,\|\phi\|_2^2) \tag{2}$$

valid $\|\phi\|_\infty < \varepsilon(T)$, where H_t is the integral operator

$$H_t\phi(x) := \int_{S^2}\int_0^t p_{t-s}(x,y)\,p_s(x,y)\,\phi(y)\,ds\,dy.$$

The first observation here is that for area-preserving deformations of the standard metric the first order term vanishes. This follows by imposing the constraint $\int_{S^2}\phi = 0$, and using that the heat kernel on S^2 is constant on the diagonal at any given $t > 0$, by the transitivity of the group of rotations. Thus, the standard metric being a critical point for the trace, a requirement for a local minimum is that the second variation be at least positive semi-definite, that is $\partial_t^2(H_t\phi,\phi) \geqslant 0$. The first part of the paper (Sections 2, 3) is concerned in proving just that; we will compute explicitly the action of H_t on spherical harmonics as sums involving the so-called Clebsch–Gordan coefficients (integrals of triple products of Legendre polynomials). These, in turn, will be evaluated using a result of Askey [A] about certain special hypergeometric series ${}_7F_6$. As a consequence of this computation we will find qualitative estimates for the second variation in terms of fractional derivatives of ϕ, as explained in the theorem below.

Throughout the paper we shall adopt the following notation: Y_ℓ a generic spherical harmonic of degree ℓ and $\|Y_\ell\|_2 = 1$; $\mathscr{H}_\ell$ the $(2\ell+1)$-dimensional space of ℓth spherical harmonics; $\lambda_\ell = \ell(\ell+1)$ the eigenvalues of Δ corresponding to $\mathcal{H}_\ell$. For $\phi = \sum_1^\infty c_\ell Y_\ell$ define $\Delta^\alpha \phi = \sum_1^\infty c_\ell (-\lambda_\ell)^\alpha Y_\ell$, $\alpha \in \mathbf{R}$, and the projections

$$P_t \phi = \sum_{\lambda_\ell t \leqslant 1} c_\ell Y_\ell, \qquad P_t^\perp \phi = \phi - P_t \phi.$$

THEOREM 2. *Let $\phi \in L^2(S^2)$, $\int_{S^2} \phi = 0$. Then, $\partial_t^2 (H_t \phi, \phi) = 0$, $\forall \phi \in \mathcal{H}_1$ and $\forall t > 0$. If $\phi \in \mathcal{H}_1^\perp$ (that is, if $\int \phi x_j = 0$, $j = 1, 2, 3$) then for some universal constants $C, C' > 0$*

$$C A_t(\phi) \leqslant \partial_t^2 (H_t \phi, \phi) \leqslant C' A_t(\phi), \qquad t > 0,$$

where

$$A_t[\phi] = \begin{cases} \|\Delta P_t \phi\|_2^2 + t^{-3} \|\Delta^{-(1/2)} P_t^\perp \phi\|_2^2 & \text{if } \ 0 < t \leqslant 1 \\ e^{-2t} \|\Delta^{-(1/2)} \phi\|_2^2 & \text{if } \ t \geqslant 1, \phi \in \mathcal{H}_2^\perp \\ t e^{-2t} \|\Delta^{-(1/2)} \phi\|_2^2 & \text{if } \ t \geqslant 1, \phi \notin \mathcal{H}_2^\perp. \end{cases}$$

Thus, the second variation of the trace is positive definite except along the directions $\phi \in \mathcal{H}_1$, where it vanishes. This last fact is a consequence of the invariance of the trace under the one-parameter family of conformal transformations of the form $|\tau'_\lambda|\, g$, where in stereographic coordinates $\tau_\lambda(z) = \lambda z$, with $\lambda > 0$ (see also [R], p. 79).

What we really need for the proof of Theorem 1, however, is the following immediate consequence of Theorem 2:

COROLLARY 1. *There exists a universal constant $C > 0$ such that for $\phi \in \mathcal{H}_1^\perp$*

$$\partial_t^2 (H_t \phi, \phi) \geqslant C \|\Delta^{-(1/2)} \phi\|_2^2, \qquad \forall t > 0.$$

We observe that this estimate is accurate only for t bounded away from zero; indeed $A_t(\phi) \to \|\Delta \phi\|_2^2$, as $t \downarrow 0$.

Theorem 2 guarantees a local minimum for the trace only along 1-parameter families of metrics emanating from g along directions in $\mathcal{H}_1^\perp$. In the second part of the paper (Section 4) we will improve the error in (2) so that at least for $w - 1 \in \mathcal{H}_1^\perp$ we can find a full L^∞ neighborhood of $w = 1$ where the trace inequality holds. More precisely will shall prove the following.

THEOREM 3. *Let $\phi \in C^\infty(S^2)$, with $1 - \phi > 0$ and $\int_{S^2} \phi = 0$. For any $T > 0$ there is a $C(T)$ such that for $0 < t \leqslant T$*

$$\left| Tr(e^{t\Delta_{1-\phi}}) - Tr(e^{t\Delta}) - \frac{t}{2} \partial_t^2 (H_t \phi, \phi) \right| \leqslant t^{-2} C(T) \|\phi\|_\infty \|\Delta^{-(1/2)} \phi\|_2^2.$$

To prove this theorem we will go back to the series expansion $Tr(e^{t\Delta_w}) - Tr(e^{t\Delta}) = \sum_2^\infty (1/k!)\, I_t^k(w-1)$, using the explicit formulas for the k-linear forms I_t^k derived in [M1]. We will prove the estimate $|I_t^k \phi| \leqslant t^{-2} C(T) \times \|\phi\|_\infty^{k-2} \|\Delta^{-(1/2)}\phi\|_2^2$ by viewing I_t^k as a multilinear singular integral, and by applying a version of the T1-Theorem of David and Journé. This last result is stated precisely, and in a more general setting, in Theorem 10, Section 4 and could be of independent interest.

Theorem 1 for the case $w-1 \in \mathscr{H}_1^\perp$ is a consequence of Corollary 1 and Theorem 2. In Section 5 we will show how to obtain the general case by a "center of mass" argument, in the same spirit as in [H, O].

Finally, we mention that a version of the trace inequality (1) for bounded domains in $\mathbf{R}^2$ with Dirichlet boundary condition, has been given by Luttinger [L]. In this case the inequality is reversed: among all bounded domains with given area, the disk has maximum trace. A by-product of this for $t \downarrow 0$ is the classical isoperimetric inequality, and for $t \uparrow \infty$ is the Faber–Krahn inequality $\lambda_1(\Omega) \geqslant \lambda_1(\Omega_0)$, Ω_0 a disk and $|\Omega| = |\Omega_0|$.

We also mention the following result by H. Montgomery [Mo] on flat tori: Among all flat tori of area 1, the one corresponding to the equilateral lattice $\langle 1, (1+i\sqrt{3})/2 \rangle$ has minimum trace.

2. COMPUTATION OF H_t

This section is devoted to the explicit computation of the bilinear form (H_t, ϕ, ϕ), on the 2-dimensional round sphere.

Let $\{Y_{j,h}\}_{h=1}^{2j+1}$ be a real-valued o.n. basis of eigenfunctions of Δ, for the $2j+1$-dimensional space $\mathscr{H}_j$. The eigenvalue corresponding to $\mathscr{H}_j$ is $j(j+1)$ and the heat kernel can be written as

$$p_t(x, y) = \sum_{j=0}^{\infty} e^{-j(j+1)t} \sum_{h=0}^{2j+1} Y_{j,h}(x)\, Y_{j,h}(y).$$

In particular, $Tr(e^{t\Delta}) = \sum_0^\infty (2j+1)\, e^{-j(j+1)t}$. Using known properties of the zonal harmonics (see [SW], p. 149) we can rewrite p_t as

$$p_t(x, y) = \frac{1}{4\pi} \sum_{j=0}^{\infty} e^{-j(j+1)t} (2j+1)\, P_j(x \cdot y).$$

See also Terras [Ter] p. 106.

Whenever convenient, for the rest of this section we will use the notation

$$\lambda_k = k(k+1).$$

Now let $\phi \in L^2(S^2)$ be real-valued. Decompose ϕ into spherical harmonics as

$$\phi = \sum_{\ell=0}^{\infty} c_\ell Y_\ell, \qquad \int_{S^2} Y_\ell^2 = 1.$$

In our future applications ϕ will actually be C^∞ and with mean 0. By definition, H_t has kernel

$$\int_0^t p_{t-s}(x, y)\, p_s(x, y)\, ds.$$

Since $\{P_k \sqrt{(2k+1)/2}\}_0^\infty$ form an o.n. basis in $L^2[-1, 1]$ we can expand

$$P_j P_k = \sum_{\ell=0}^{\infty} c(j, k, \ell)\, P_\ell$$

where the sum is actually finite, since $P_j P_k$ is a polynomial. Also we will see below that

$$c(j, k, \ell) = \frac{2\ell+1}{2} \int_{-1}^{1} P_j P_k P_\ell \geqslant 0.$$

Thus,

$$p_{t-s}(x, y)\, p_s(x, y) = \frac{1}{(4\pi)^2} \sum_{\ell, j, k=0}^{\infty} e^{-\lambda_j (t-s)\, \lambda_k s} c(j, k, \ell) \times (2k+1)(2j+1)\, P_\ell(x \cdot y).$$

But

$$\begin{aligned} &\frac{2\ell+1}{4\pi} \int_{S^2} \int_{S^2} P_\ell(x \cdot y)\, \phi(x)\, \phi(y)\, dx\, dy \\ &= \frac{2\ell+1}{4\pi} c_\ell^2 \int_{S^2} \int_{S^2} P_\ell(x \cdot y)\, Y_\ell(x)\, Y_\ell(y)\, dx\, dy \\ &= c_\ell^2 \int_{S^2} \int_{S^2} \sum_{h=0}^{2\ell+1} Y_{\ell,h}(x)\, Y_{\ell,h}(y)\, Y_\ell(x)\, Y_\ell(y)\, dx\, dy = c_\ell^2. \end{aligned}$$

After integrating in s (see Remark 2 below) we finally get the decomposition

$$(H_t \phi, \phi) = \sum_{\ell=0}^{\infty} c_\ell^2 (H_t Y_\ell, Y_\ell), \tag{3}$$

where

$$(H_t Y_\ell, Y_\ell) = \frac{1}{4\pi} \sum_{k=0}^{\infty} (2k+1)[B_\ell(k) + tA_\ell(k)]\, e^{-\lambda_k t} \tag{4}$$

and

$$\begin{cases} A_\ell(k) = \dfrac{2k+1}{2} \displaystyle\int_{-1}^{1} P_k^2 P_\ell \\ B_\ell(k) = \displaystyle\sum_{j \neq k} \frac{2j+1}{j(j+1) - k(k+1)} \int_{-1}^{1} P_j P_k P_\ell. \end{cases} \tag{5}$$

Note that $(H_t Y,$ is the same for all $Y \in \mathscr{H}_\ell$ with $\|Y\|_2 = 1$, and $\ell \geqslant 0$.

Remark 2. The functions $A_\ell(k)$ and $B_\ell(k)$ are bounded above independently of ℓ and k. This can be seen directly from (3). Using the well known fact that $|P_\ell| \leqslant 1$ we get immediately that $|A_\ell| \leqslant 1$. Also, by Hölder's inequality, $|\int_{-1}^{1} P_j P_k P_\ell| \leqslant |\int_{-1}^{1} P_j P_k| \leqslant C[(2j+1)(2k+1))]^{-(1/2)}$, which implies

$$|B_\ell(k)| \leqslant \frac{C}{\sqrt{k+1}} \sum_{j \neq k} \frac{1}{|j-k|\, \sqrt{j+1}}$$

which tends to 0 as $k \to \infty$, as the reader can easily check. This observation legitimates the various operations performed to obtain (3)–(5).

The following theorem gives explicit expressions for A_ℓ and B_ℓ.

THEOREM 4. *For $\phi = \sum_1^\infty c_\ell Y_\ell$ we have $(H_t, \phi, \phi) = \sum_1^\infty c_\ell^2 (H_t Y_\ell, Y_\ell)$, where*

$$(H_t Y_\ell, Y_\ell) = \frac{1}{4\pi} \sum_{k=0}^{\infty} (2k+1)[B_\ell(k) + tA_\ell(k)]\, e^{-\lambda_k t}$$

and the coefficients $A_\ell(k)$ and $B_\ell(k)$, $\ell \geqslant 1$ are the following nonnegative rational functions of $k \in \{0, 1, 2...\}$:

$$\begin{cases} A_{2p+1}(k) = 0 & \text{if } \; k \geqslant 0 \\ A_{2p}(k) = 0 & \text{if } \; 0 \leqslant k \leqslant p-1 \\ A_{2p}(k) = (2k+1)\, \dfrac{\Gamma(p+1/2)^2\, \Gamma(k-p+1/2)(k+p)!}{2\pi p!^2 \Gamma(k+p+3/2)(k-p)!} & \text{if } \; k \geqslant p \geqslant 0 \end{cases}$$

$$\begin{cases} B_{2p+1}(k) = \dfrac{p!^2 \Gamma(p-k+1/2)\,\Gamma(p+k+3/2)}{2\Gamma(p+3/2)^2\,(p-k)!(p+k+1)!} & \textit{if} \quad 0 \leqslant k \leqslant p \\ B_{2p+1}(k) = 0 & \textit{if} \quad k \geqslant p+1 \\ B_{2p}(k) = \dfrac{\Gamma(p+1/2)^2\,(p-k-1)!\,(p+k)!}{2p!^2 \Gamma(p-k+1/2)\,\Gamma(p+k+3/2)} & \textit{if} \quad 0 \leqslant k \leqslant p-1 \\ B_{2p}(k) = \dfrac{2A_{2p}(k)}{2k+1} \displaystyle\int_0^1 t^{2k} \left(\frac{t^{-2p}}{1+t} + \frac{t^{2p+1}}{1+t} - 1 \right) dt & \textit{if} \quad k \geqslant p \geqslant 0. \end{cases}$$

Remark 3. Observe that $A_0 \equiv 1$ and $B_0 \equiv 0$.

Proof. We start with the linearization formula (see Askey [A])

$$P_k P_\ell = \sum_{n=0}^{\ell \wedge k} b_{\ell, k, n} P_{\ell+k-2n},$$

where $\ell \wedge k = \min(\ell, k)$ and

$$b_{\ell, k, n} = \frac{(1/2)_{\ell-n}\,(1/2)_{k-n}\,(1/2)_n\,(\ell+k-n)!(\ell+k-2n+1/2)}{(1/2)_{\ell+k-n}\,(\ell-n)!\,(k-n)!\,n!(\ell+k-n+1/2)},$$

and where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the shifted factorial. The coefficients $b_{\ell, k, n}$ are defined for $n \leqslant \ell \wedge k$, however we define them to be 0 for $n > \ell \wedge k$.

By the orthogonality relation for P_j we find

$$\int_{-1}^1 P_j P_k P_\ell = \frac{2}{2j+1} \sum_{n=0}^{\ell \wedge k} b_{\ell, k, n} \delta_{\ell+k-2n, j} \geqslant 0,$$

with equality $\Leftrightarrow j+k+\ell$ is odd or $j \notin [\,|\ell-k|, \ell+k]$. This already implies the statements for A_ℓ. In particular note that, when $k \geqslant p$

$$A_{2p}(k) = b_{2p, k, p}.$$

From (5) we obtain

$$B_\ell(k) = \sum_{\substack{n=0 \\ 2n \neq \ell}}^{\ell \wedge k} \frac{2b_{\ell, k, n}}{(\ell-2n)(\ell-2n+2k+1)}.$$

Define the sum

$$B_\ell^z(k) = \sum_{n=0}^{\ell \wedge k} \frac{2b_{\ell, k, n}}{(\ell-2n+2z)(\ell-2n-2z+2k+1)},$$

a meromorphic function of $z \in \mathbf{C}$. Luckily, Askey [A] calculated $B_\ell^z(k)$ by rewriting the sum as a very well posed 2-balanced hypergeometric series ${}_7F_6$ and then by applying Dougall's formula for such series (see Remark 4 below). The result is

$$B_\ell^z(k) = \frac{\Gamma(z+1/2(1+\ell))\,\Gamma(z-k+1/2(1-\ell))\,\Gamma(z-k+\ell/2)\,\Gamma(z-\ell/2)}{\begin{pmatrix}(\ell+2k-2z+1)\,\Gamma(z+1+\ell/2)\,\Gamma(z+1/2(1-\ell))\\ \times\Gamma(z-k+1/2(1+\ell))\,\Gamma(z-k-\ell/2)\end{pmatrix}}$$

an appropriate limit being taken wherever the Γ function has a pole. Consider first the case ℓ is odd, $\ell = 2p+1$. We have that

$$B_{2p+1}(k) = \lim_{z\to 0} B_{2p+1}^z(k).$$

When $k \geqslant p+1$, the denominator in the above expression for B_ℓ^z has two simple poles at $z=0$, and the numerator has one, so that $B_{2p+1}(k)=0$. When $0 \leqslant k \leqslant p$, both numerator and denominator have a simple pole at $z=0$, and $\Gamma(z-p-k)/\Gamma(z-p) \sim (-1)^k\, p!/(p+k)!$.

If, on the other hand, ℓ is even, we can still recover B_ℓ since, for z not an integer

$$B_\ell^z(k) = \sum_{\substack{n=0\\ 2n\neq\ell}}^{\ell\wedge k} (\cdots) + \frac{b_{\ell,k,\ell/2}}{z(2k-2z+1)},$$

so that

$$B_\ell(k) = \lim_{z\to 0}\left(B_\ell^z(k) - \frac{b_{\ell,k,\ell/2}}{z(2k+1)}\right) - \frac{2b_{\ell,k,\ell/2}}{(2k+1)^2}. \tag{6}$$

Recall that $b_{\ell,k,\ell/2} = 0$ for $k < \ell/2$.

For $\ell = 2p$,

$$B_{2p}^z(k) = \frac{\Gamma(z+p+1/2)\,\Gamma(z-p-k+1/2)\,\Gamma(z+p-k)\,\Gamma(z-p)}{\begin{pmatrix}(2p+2k-2z+1)\,\Gamma(z+p+1)\,\Gamma(z-p+1/2)\\ \times\Gamma(z+p-k+1/2)\,\Gamma(z-p-k)\end{pmatrix}}.$$

When $0 \leqslant k \leqslant p-1$, numerator and denominator have one simple pole at 0 and we find the desired formula as in the odd case. When $k \geqslant p$ the numerator has two simple poles at 0 and the denominator has only one, so we need to take in to account the contributions of all factors. Using known properties of the Gamma function we find

$$\frac{\Gamma(z+p-k)\,\Gamma(z-p)}{\Gamma(z-p-k)} = \frac{(-1)^p\,(p+k)!}{(k-p)!\,p!}\left[\frac{1}{z} + A + O(z)\right]$$

$$\frac{\Gamma(z+p+1/2)}{\Gamma(z-p+1/2)} = \frac{(-1)^p}{\pi}\,\Gamma\left(p+\frac{1}{2}\right)^2\,[1+O(z^2)]$$

$$\frac{\Gamma(z-p-k+1/2)}{\Gamma(z+p-k+1/2)} = \frac{\Gamma(k-p+1/2)}{\Gamma(k+p+1/2)}\,[1+Bz+O(z^2)]$$

$$\frac{1}{\Gamma(z+p+1)} = \frac{1}{p!}\,[1-Cz+O(z^2)]$$

$$\frac{1}{2p+2k-2z+1} = \frac{1}{2p+2k+1}\,[1+Dz+O(z^2)],$$

where A, B, C, D are given in terms of the function $\psi = \Gamma'/\Gamma$ by

$$A = \psi(p+1) + \psi(k-p+1) - \psi(k+p+1)$$

$$B = \psi\left(k+p+\frac{1}{2}\right) - \psi\left(k-p+\frac{1}{2}\right)$$

$$C = \psi(p+1)$$

$$D = \frac{2}{2p+2k+1}.$$

Putting all this together and using (6) yields, after a few computations,

$$\begin{aligned} B_{2p}(k) &= \frac{2b_{2p,\,k,\,p}}{2k+1}\left[\frac{A+B-C+D}{2} - \frac{1}{2k+1}\right] \\ &= \frac{2b_{2p,\,k,\,p}}{2k+1}\left[\frac{G(2k-2p+1) - G(2k+2p+1)}{2}\right. \\ &\qquad \left. - \frac{1}{2k+1} + \frac{1}{2k+2p+1}\right], \end{aligned}$$

where G is the G-function defined by $G(2z) = \psi(\frac{1}{2}+z) - \psi(z)$. By using the relation $G(1+n) = 2\int_0^1 t^n(1+t)^{-1}\,dt$, n integer, $n \geqslant -1$ ([Ba], p. 20)

$$\begin{aligned} B_{2p}(k) &= \frac{2b_{2p,\,k,\,p}}{2k+1}\int_0^1 \left(\frac{t^{2k-2p} - t^{2k+2p}}{1+t} - t^{2k} + t^{2k+2p}\right) dt \\ &= \frac{2b_{2p,\,k,\,p}}{2k+1}\int_0^1 t^{2k}\left(\frac{t^{-2p}}{1+t} + \frac{t^{2p+1}}{1+t} - 1\right) dt, \end{aligned}$$

which is the stated expression. Notice that the convexity of t^{2p} implies that the integrand is positive in $(0, 1)$, and so then is B_{2p}. This completes the proof of Theorem 4. ∎

Remark 4. The identity $B_\ell(k)=0$ for ℓ odd, and $k \geqslant (\ell+1)/2$ appeared first in the paper by Din [Din], in connections with stability properties of some classical solutions of the $O(n)$ nonlinear σ-model in 2 dimensions. Din reduced it to showing that a certain integral involving products of Legendre polynomials and Legendre functions of the second kind is zero. Askey's paper is actually concerned with giving formulas for more general such integrals, but, as he remarks on page 302 of [A], his method works directly also for the sums defining our B_ℓ^z for ℓ odd.

3. PROOF OF THEOREM 2

The first identity in Theorem 4 reduces Theorem 2 to the case $\phi = Y_\ell$. However, by examining the structure of A_ℓ and B_ℓ, we can already guess that the most delicate case is going to be when ℓ is even. We are now going to work out some more the case ℓ even, then prove Theorem 3.

For convenience we normalize A_ℓ, B_ℓ as follows:

$$F_p := \frac{\pi p!^2}{\Gamma(p+1/2)^2} A_{2p}, \qquad H_p := \frac{\pi p!^2}{\Gamma(p+1/2)^2} B_{2p}.$$

From Theorem 4 and $\Gamma(1+z) = z\Gamma(z)$ we obtain the induction formula

$$F_{p+1}(k) = \frac{(k-p)(k+p+1)}{(k+p+3/2)(k-p-1/2)} F_p(k),$$

which holds for all $k \geqslant 0$, $p \geqslant 0$. For $p, k \geqslant 0$ set

$$q_k = q_k(p) = \frac{1}{(k+p+3/2)(k-p-1/2)}$$

so that the induction formula can be rewritten as

$$F_{p+1}(k) = [1 + (p + \tfrac{3}{4})\, q_k]\, F_p(k) \tag{7}$$

Lemma 1. *If* $p \geqslant 0$, $k \geqslant p$, *then*

$$B_{2p}(k) = -\frac{1}{2k+1} \frac{d}{dk} A_{2p}(k).$$

Proof. By induction on p we show that $-\partial_k F_p(k) = (2k+1)\, H_p(k)$. This is obvious when $p=0$. From Theorem 4 we have $(2k+1)\, H_p = 2F_p \Psi_p$, where

$$\Psi_p(k) = \int_0^1 t^{2k}(1+t)^{-1}\,(t^{-2p} + t^{2p+1} - 1 - t)\, dt.$$

Suppose that $-\partial_k F_p = 2F_p \Psi_p$, then by the induction formula (7)

$$\begin{aligned}
-\partial_k F_{p+1} &= -\left(p+\frac{3}{4}\right)\partial_k q_k F_p - \left[1+\left(p+\frac{3}{4}\right) q_k\right]\partial_k F_p \\
&= \frac{(p+3/4)(2k+1)}{(k+p+3/2)^2(k-p-1/2)^2} F_p + 2\Psi_p F_{p+1} \\
&= 2F_{p+1}\left[\frac{2(4p+3)(2k+1)}{(2k-2p)(2k+2p+2)(2k+2p+3)(2k-2p-1)} + \Psi_p\right].
\end{aligned}$$

So it will be enough to verify the identity

$$\Psi_{p+1} - \Psi_p = \frac{2(4p+3)(2k+1)}{(2k-2p)(2k+2p+2)(2k+2p+3)(2k-2p-1)},$$

which can be checked by calculating

$$\begin{aligned}
\Psi_{p+1} - \Psi_p &= \int_0^1 t^{2k}(1+t)^{-1}\,(t^{-2p-2} + t^{2p+3} - t^{-2p} - t^{2p+1})\, dt \\
&= \int_0^1 t^{2k}(1-t)(t^{-2p-2} - t^{2p+1})\, dt. \quad \blacksquare
\end{aligned}$$

The following theorem is a key element in the proof of Theorem 2.

THEOREM 5. *For* $p \geqslant 0$, $N \geqslant \max(p, 1)$, *the function*

$$Q_p^N(t) = \sum_{k=N}^{\infty} (2k+1)[B_{2p}(k) + tA_{2p}(k)]\, e^{-\lambda_k t}$$

is strictly convex and decreasing in $(0, \infty)$.

Proof. Define

$$\tilde{Q}_p^N(t) = \sum_{k=N}^{\infty} (2k+1)[H_p + tF_p]\, e^{-\lambda_k - t} = \frac{\pi p!^2}{\Gamma(p+1/2)^2}\, Q_p^N(t).$$

First of all, notice that $\tilde{Q}_p^N(t) > 0$, and $\tilde{Q}_p^N(t) \to 0$ as $t \to \infty$. Thus, it is enough to show that $\partial_t^2 \tilde{Q}_p^N > 0$, in $(0, \infty)$. We do this by induction on p.

By Remark 2 we can differentiate under the sums

$$\partial_t \tilde{Q}_p^N(t) = \sum_{k=N}^{\infty} (2k+1)[-\lambda_k H_p + F_p - t\lambda_k F_p]\, e^{-\lambda_k t}$$

$$\partial_t^2 \tilde{Q}_p^N(t) = \sum_{k=N}^{\infty} (2k+1)[\lambda_k^2 H_p - 2\lambda_k F_p + t\lambda_k^2 F_p]\, e^{-\lambda_k t}.$$

Assume that for some $p \geqslant 0$ we have $\partial_t^2 \tilde{Q}_p^N > 0$, $\partial_t \tilde{Q}_p^N < 0$, $\forall t > 0$, $\forall N \geqslant \max(p, 1)$. By Lemma 1 and the induction formula for F_p we get, for all $k \geqslant p+1$

$$H_{p+1}(k) = -\frac{\partial_k F_{p+1}(k)}{2k+1} \left[1 + \left(p + \frac{3}{4}\right) q_k \right] H_p + \left(p + \frac{3}{4}\right) q_k^2 F_p(k).$$

Thus, $\forall N \geqslant p+1$ we find that

$$\begin{aligned}\partial_t^2 \tilde{Q}_{p+1}^N(t) = &\sum_{k=N}^{\infty} (2k+1)[\lambda_k^2 H_p - 2\lambda_k F_p + t\lambda_k^2 F_p]\, e^{-\lambda_k t} \\ &+ \sum_{k=N}^{\infty} (2k+1)(p+\tfrac{3}{4})[\lambda_k^2 q_k H_p + \lambda_k^2 q_k^2 F_p \\ &- 2\lambda_k q_k F_p + t\lambda_k^2 q_k F_p]\, e^{-\lambda_k t}.\end{aligned}$$

Now, one checks that $\lambda_k q_k = 1 + (p^2 + 2p + \frac{3}{4})\, q_k$ so that

$$\begin{aligned}\partial_t^2 \tilde{Q}_{p+1}^N(t) &= \partial_t^2 \tilde{Q}_p^N(t) + (p+\tfrac{3}{4}) \sum_{k=N}^{\infty} (2k+1)\{[\lambda_k + (p^2+2p+\tfrac{3}{4})\, \lambda_k q_k]\, H_p \\ &\quad + [(p^2+2p+\tfrac{3}{4})^2\, q_k^2 + t\lambda_k - 1 + t(p^2+2p+\tfrac{3}{4})\, \lambda_k q_k]\, F_p\}\, e^{-\lambda_k t} \\ &= \partial_t^2 \tilde{Q}_p^N(t) - (p+\tfrac{3}{4})\, \partial_t \tilde{Q}_p^N(t) + (p+\tfrac{3}{4}) \sum_{k=N}^{\infty} (2k+1) \\ &\quad \times [\lambda_k q_k\, (p^2+2p+\tfrac{3}{4})(H_p + tF_p) + (p^2+2p+\tfrac{3}{4})^2\, q_k^2 F_p]\, e^{-\lambda_k t}.\end{aligned} \tag{8}$$

Therefore, by our induction hypothesis, and because $q_k(p) > 0$ for $k \geqslant p+1$, we obtain $\partial_t^2 \tilde{Q}_{p+1}^N(t) > 0$.

We must show now that, for all $t > 0$

$$\partial_t^2 \tilde{Q}_0^N(t) = \sum_{k=N}^{\infty} (2k+1)(t\lambda_k^2 - 2\lambda_k)\, e^{-\lambda_k t} > 0.$$

We claim that it is enough to show that $\partial_t^2 \tilde{Q}_0^1 > 0$. In fact, suppose that for some $N \geqslant 1$ we have $\partial_t^2 \tilde{Q}_0^N(t) > 0$, $\partial t > 0$. Then,

$$\partial_t^2 \tilde{Q}_0^{N+1}(t) = \partial_t^2 \tilde{Q}_0^N(t) - (2N+1)(t\lambda_N^2 - 2\lambda_N)\, e^{-\lambda_N t} > 0, \qquad \forall t < \frac{2}{\lambda_N}.$$

But also

$$\partial_t^2 \tilde{Q}_0^{N+1}(t) = \sum_{k=N+1}^{\infty} (2k+1)(t\lambda_k^2 - 2\lambda_k)\, e^{-\lambda_k t} > 0, \qquad \forall t > \frac{2}{\lambda_{N+1}}.$$

Since $\lambda_k = k(k+1)$ is increasing, we get $\partial_t^2 \tilde{Q}_0^{N+1}(t) > 0$, $\forall t > 0$.

We are left to prove the non-trivial inequality:

$$\partial_t^2 \tilde{Q}_0^1(t) = \sum_{k=1}^{\infty} (2k+1)(t\lambda_k^2 - 2\lambda_k)\, e^{-\lambda_k t} > 0, \qquad \forall t > 0.$$

First of all, notice that $\lambda_1 = 2$ so that the sum is positive when $t \geqslant 1$. In fact we can take $t \geqslant 7/10$, since

$$\partial_t^2 \tilde{Q}_0^1(t) = 12e^{-6t}[e^{4t}(t-1) + 15t - 5] + \sum_{k=3}^{\infty} (2k+1)(t\lambda_k^2 - 2\lambda_k)\, e^{-\lambda_k t}$$

and the function in brackets and all the terms of the series are positive when $t \geqslant 7/10$.

To treat the case $t \in [0, 7/10]$, we find an asymptotic expansion for $\partial_t^2 \tilde{Q}_0^1(t)$, with precise control on the error. We use a related result of Mulholland [Mu].

Define

$$\zeta(t) = \sum_{k=0}^{\infty} (2k+1)\, e^{-(k+1/2)^2 t}.$$

In [Mu] Mulholland showed that

$$\zeta(t) = \frac{1}{t} + \zeta_0(t)$$

where ζ_0 satisfies

$$\begin{cases} \zeta_0^{(k)}(0) = a_k := (-1)^k\, (k+1)^{-1}\, (1 - 2^{-2k-1})\, B_{2k+2} \\ |\zeta_0^{(k)}(t)| \leqslant E_k := (-1)^k\, 2^{k+1} (k+1)^{-1}\, B_{2k+2} \end{cases}$$

and where the B's are the Bernoulli numbers $B_{2k}=2(-1)^{k+1}(2k)! \times \sum_1^\infty (2\pi m)^{-2k}$. Thus, for $t\in[0,1]$

$$\zeta_0^{(i)}(t)=\sum_{k=0}^{N-1}\frac{t^k}{k!}a_{k+i}+\frac{t^N}{N!}\zeta_0^{(N+i)}(\theta_N^i t), \qquad \theta_N^i\in[0,1].$$

Recall that $B_2=\frac{1}{6}$, $B_4=-\frac{1}{30}$, $B_6=\frac{1}{42}$, $B_8=-\frac{1}{30}$ (see [Ba]). We note that for the very first few terms the error bound E_k improves as k increases, although it gets worse as k gets large (since $B_{2k}\sim C(2k)!\,(2\pi)^{-2k}/(2k)$).

Now, $\tilde{Q}_0^1$ and ζ are related via the identity

$$\tilde{Q}_0^1(t)=te^{t/4}\zeta(t)-t.$$

Set $\sigma(t)=t\zeta(t)$. Then

$$\partial_t^2\tilde{Q}_0^1=e^{t/4}(\sigma''+\tfrac{1}{2}\sigma'+\tfrac{1}{16}\sigma),$$

so it is enough to show that $\sigma''+\frac{1}{2}\sigma'+\frac{1}{16}\sigma>0$, when $t\in(0,7/10]$. From the above expansions we derive

$$\begin{aligned}(\sigma-1)^{(i)}(t)=t\zeta_0^{(i)}(t)+i\zeta_0^{(i-1)}(t)=ia_{i-1}&+\sum_{k=1}^{N-1}\frac{t^k}{k!}(k+i)\,a_{k+i-1}\\ &+\frac{t^N}{N!}\left[N\zeta_0^{(N+i-1)}(\theta_{N-1}^i t)+i\zeta_0^{(N+i-1)}(\theta_N^{i-1}t)\right].\end{aligned}$$

In particular,

$$\begin{aligned}\sigma(t)&=1+ta_0+t^2R_1(t)\\ \sigma'(t)&=a_0+2ta_1+t^2R_2(t)\\ \sigma''(t)&=2a_1+3ta_2+t^2R_3(t)\end{aligned}$$

with $a_0=\frac{1}{12}$, $a_1=\frac{7}{480}$, $a_2=\frac{31}{4032}$, and $|R_1(t)|\leqslant\frac{1}{15}$, $|R_2(t)|\leqslant\frac{2}{21}$, $|R_3(t)|\leqslant\frac{4}{15}$. This implies that

$$\sigma''+\frac{1}{2}\sigma'+\frac{1}{16}\sigma\geqslant\frac{2}{15}+\frac{3}{70}t-\frac{107}{336}t^2$$

which is positive for $0\leqslant t\leqslant(36+4\sqrt{7571})/535=0.717843...$, and in particular for $0\leqslant t\leqslant 7/10$. ∎

Remark 5. Theorem 5 in the case $p=0$, $N=1$, says that on the standard S^2 the function

$$t[Tr(e^{t\Delta})-1]=t\sum_{k=1}^{\infty}(2k+1)\,e^{-k(k+1)t}$$

is strictly convex and decreasing, for $t \geqslant 0$. By integrating twice $\partial_t^2(t\,Tr(e^{t\Delta}))$ and using the expansion $t\,Tr(e^{t\Delta}) = 1 + t/3 + O(t^2)$ we obtain the following bounds for the trace in S^2

$$\frac{1}{t} + \frac{1}{3} < Tr(e^{t\Delta}) < \frac{1}{t} + 1 \qquad \forall t > 0.$$

Proof of Theorem 2. Let $\phi = \sum_1^\infty c_\ell Y_\ell$, $\int Y_\ell^2 = 1$. The first statement of the theorem follows from Theorem 4, since $(H_t Y_1, Y_1) \equiv (4\pi)^{-1}$.

Case I: $0 < t \leqslant 1$. Let us first prove that

$$\partial_t^2(H_t\phi, \phi) \geqslant C \sum_{\substack{\lambda_\ell t < 1 \\ \ell \geqslant 2}} c_\ell^2 \lambda_\ell^2 + \frac{C}{t^3} \sum_{\substack{\lambda_\ell t \geqslant 1 \\ \ell \geqslant 2}} \frac{c_\ell^2}{\lambda_\ell},$$

for some $C > 0$. It is enough to show that

$$\partial_t^2(H_t Y_\ell, Y_\ell) \geqslant C \begin{cases} \lambda_\ell^2 & \text{if} \quad \lambda_\ell t < 1 \\ t^{-3}\lambda_\ell^{-1} & \text{if} \quad \lambda_\ell \geqslant 1. \end{cases} \tag{9}$$

Let ℓ be odd, say $\ell = 2p + 1$. From Theorem 4 we get

$$(H_t Y_\ell, Y_\ell) = \frac{1}{4\pi} \sum_{k=0}^{p} (2k+1)\, B_{2p+1}(k)\, e^{-\lambda_k t},$$

where

$$B_{2p+1}(k) = \frac{\Gamma(p+1)^2\, \Gamma(p-k+1/2)\, \Gamma(p+k+3/2)}{2\Gamma(p+3/2)^2\, \Gamma(p-k+1)\, \Gamma(p+k+2)}.$$

By the Stirling asymptotics of the gamma function ([Ba], p. 47) we know that $\Gamma(z)/\Gamma(z+\alpha) \sim z^{-\alpha}$, as $|z| \to \infty$. Therefore,

$$Cn^{-\alpha} \leqslant \frac{\Gamma(n)}{\Gamma(n+\alpha)} \leqslant C' n^{-\alpha},$$

for all integers $n \geqslant 1$, where C, C' depends on α. It follows that

$$C \leqslant \frac{B_{2p+1}(k)}{p^{-1}(p+k)^{-1/2}\,(p-k+1)^{-1/2}} \leqslant C', \tag{10}$$

for all $p \geqslant 1$, $0 \leqslant k \leqslant p$, with C, C' independent of k, p. In particular,

$$\partial_t^2(H_t Y_\ell, Y_\ell) = \frac{1}{4\pi} \sum_{k=1}^{p} (2k+1)\, \lambda_k^2 B_{2p+1}(k)\, e^{-\lambda_k t}$$

$$\geqslant \frac{C}{\lambda_p} \sum_{k=1}^{p} (2k+1)\, \lambda_k^2 e^{-\lambda_k t}. \tag{11}$$

Observe that $\frac{2}{15}\lambda_\ell \leqslant \lambda_p \leqslant \frac{1}{4}\lambda_\ell$.

If $\lambda_\ell t < 1$, then

$$\partial_t^2(H_t Y_\ell, Y_\ell) \geqslant \frac{C}{\lambda_p} \sum_{k=1}^{p} k^5 \geqslant \frac{C}{\lambda_p} p^6 \geqslant C\lambda_\ell^2. \tag{12}$$

If $\lambda_\ell t \geqslant 1$, choose N so that $N < p$ and $\lambda_N t \geqslant 10^{-2}$. The function $f(x) = x^2 e^{-xt}$ is increasing in $[0, 2/t]$, so that $f(x(x+1))(2x+1)$ is increasing in $[0, N]$ and

$$\partial_t^2(H_t, Y_\ell, Y_\ell) \geqslant \frac{C}{\lambda_p} \sum_{k=1}^{N} \int_{k-1}^{k} f(\lambda_k)(2k+1)\, dx$$

$$\geqslant \frac{C}{\lambda_p} \int_0^N f(x(x+1))(2x+1)\, dx$$

$$= \frac{C}{\lambda_p} \int_0^{\lambda_N} x^2 e^{-xt}\, dx \geqslant \frac{C}{\lambda_p t^3} \int_0^{10^{-2}} y^2 e^{-y}\, dy \geqslant \frac{C}{\lambda_\ell t^3}. \tag{13}$$

Consider now the case ℓ even, say $\ell = 2p$, $p \geqslant 1$. From Theorem 4 we get

$$(H_t Y_\ell, Y_\ell) = \frac{1}{4\pi} \sum_{k=0}^{p-1} (2k+1)\, B_{2p}(k)\, e^{-\lambda_k t}$$

$$+ \frac{1}{4\pi} \sum_{k=p}^{\infty} (2k+1)[B_{2p}(k) + tA_{2p}(k)]\, e^{-\lambda_k t}. \tag{14}$$

From Theorem 5 the second sum is strictly convex.

If $p = 1$ then (9) follows since $\partial_t^2(H_t Y_2, Y_2) \geqslant C > 0$ for $t \in (0, 1]$, as one can check using Theorem 5 and formula (8). If $p > 1$, then Theorem 5 implies

$$\partial_t^2(H_t, Y_\ell, Y_\ell) \geqslant \frac{1}{4\pi} \sum_{k=0}^{p-1} (2k+1)\, \lambda_k^2 B_{2p}(k)\, e^{-\lambda_k t}$$

with

$$B_{2p}(k) = \frac{\Gamma(p+1/2)^2\,(p-k-1)!(p+k)!}{2p!^2\Gamma(p-k+1/2)\,\Gamma(p+k+3/2)}, \qquad 0 \leqslant k \leqslant p-1.$$

and (9) follows in a manner very similar to that of the case ℓ odd. This complete the proof of (a).

To prove the reverse bounds in (9), one can check that for ℓ odd (10) implies that the reverse bounds in (11), (12), (13), hold, with some different constant C'. In the case ℓ even, $\ell \geqslant 4$ (the case $\ell = 2$ is obvious) the same is true for the first sum in (14). To estimate from above the second sum of (14) observe that from Remark 2 A_{2p} and B_{2p} are bounded above, independently of k, p. Hence, for $p \geqslant 2$

$$\frac{\partial^2}{\partial t^2} \sum_{k=p}^{\infty} (2k+1)[B_{2p}(k) + tA_{2p}(k)]\, e^{-\lambda_k t} \leqslant C' e^{-\lambda_p t/2} \tag{15}$$

which satisfies the reverse bounds in (9).

Case II: $t \geqslant 1$. It is enough to prove that

$$Ce^{-2t}\lambda_\ell^{-1} \leqslant \partial_t^2(H_t Y_\ell, Y_\ell) \leqslant C'e^{-2t}\lambda_\ell^{-1} \qquad \text{if} \quad c_2 = 0;$$

$$Cte^{-2t}\lambda_\ell^{-1} \leqslant \partial_t^2(H_t, Y_\ell, Y_\ell) \leqslant C'te^{-2t}\lambda_\ell^{-1} \qquad \text{if} \quad c_2 \neq 0.$$

If $\ell = 2p+1$, from (10), (11) we deduce that

$$\frac{C}{\lambda_p} e^{-2t} \leqslant \partial_t^2(H_t Y_\ell, Y_\ell) \leqslant \frac{C'}{p^{3/2}} \sum_{k=1}^{p} \frac{(2k+1)\,\lambda_k^2}{(p-k+1)^{1/2}} e^{-\lambda_k t}$$
$$\leqslant \frac{C'}{p^{3/2}} \left(\sum_{1 \leqslant k \leqslant p/2} + \sum_{p/2 < k \leqslant p} \right).$$

Now

$$\frac{1}{p^{3/2}} \sum_{1 \leqslant k \leqslant p/2} \frac{(2k+1)\,\lambda_k^2}{(p-k+1)^{1/2}} e^{-\lambda_k t}$$
$$\leqslant \frac{1}{p^2} e^{-2t} \sum_{1 \leqslant k \leqslant p/2} (2k+1)\,\lambda_k^2 e^{-(\lambda_k - 2)} \leqslant \frac{C'}{\lambda_\ell} e^{-2t}$$

and, for $p > 1$, one finds

$$\frac{1}{p^{3/2}} \sum_{p/2 < k \leqslant p} \frac{(2k+1)\,\lambda_k^2}{(p-k+1)^{1/2}} e^{-\lambda_k t} \leqslant \frac{C'}{p^{3/2}} \sum_{p/2 < k \leqslant p} p^5 e^{-\lambda_k t} \leqslant \frac{C'}{\lambda_\ell} e^{-2t}.$$

If $\ell = 2p$, $p \geqslant 2$, we use the expression (14), in which the first sum can be treated as above and the second sum is bounded above by $C' e^{-\lambda_p t/2} \leqslant C' e^{-2t} \lambda_p^{-1}$, by (15). Finally, if $\ell = 2$, in a similar way we obtain the same bounds with an extra factor of t, since $A_2(1) \neq 0$. ∎

4. ERROR ESTIMATES AND PROOF OF THEOREM 3

The proof of Theorem 3 will be based on the perturbations results obtained in [M1], which we now recall.

For $\phi \in L^\infty(S^2)$ define iterated kernels

$$
\begin{aligned}
&\beta_t^0(x, y) = p_t(x, y), \\
&\beta_t^k(x, y) = \int_0^t \int_{S^2} p_{t-s}(x, z)\, \beta_s^{k-1}(z, y)\, \phi(z)\, dz\, ds, \qquad k \geqslant 1
\end{aligned}
\tag{16}
$$

and integral operators

$$
H_t^k f(x) = \int_{S^2} \int_0^t p_{t-s}(x, y)\, \beta_t^{k-2}(x, y)\, f(y)\, ds\, dy, \qquad k \geqslant 2. \tag{17}
$$

Note that $H_t^2 = H_t$, the second variation operator defined in Section 1. In [M1] we showed that $\partial_t^k H_t^k$ is bounded in L^2 and that the following perturbation series for the trace holds:

$$
Tr(e^{t\Delta_{1-\phi}}) - Tr(e^{t\Delta}) = t \frac{d}{dt} \int_{S^2} p_t(x, x)\, \phi(x)\, dx + \sum_{k=2}^{\infty} \frac{t}{k} \frac{d^k}{dt^k} (H_t^k \phi, \phi)
$$

valid for $0 < t \leqslant T$ and $\|\phi\|_\infty < \varepsilon = \varepsilon(T)$ small enough.

The purpose of this section is to estimate the remainder terms $\partial_t^k (H_p^k \phi, \phi)$ for $k \geqslant 3$. The main result is that for $0 < t \leqslant T$

$$
|\partial_t^k (H_t^k f, f)| \leqslant t^{-2} C(T)^{k+1}\, \|\phi\|_\infty^{k-2}\, \|\Delta^{-1} f\|_2^2 \tag{18}
$$

whenever $f \in L^2$, $\int f = 0$. Theorem 3 is an easy consequence of this bound, after noticing that for area preserving deformations $(1-\phi)\, g$ of the standard metric we have $\int_{S^2} \phi = 0$ and

$$
\int_{S^2} p_t(x, x)\, \phi(x)\, dx = 0 \qquad \forall t > 0,
$$

due to the transitivity of the group of rotations on S^2.

We shall derive (18) in a general compact n-dimensional manifold, without boundary, since the arguments involved apply directly in this generality.

Throughout this section M will denote a compact n-dimensional Riemannian manifold without boundary, with metric g and Laplacian $\Delta = \operatorname{div} \nabla$. Eigenvalues and eigenfunctions of Δ are denoted by λ_j, ϕ_j. The *complex time* heat kernel is the fundamental solution for the complex heat operator $\Delta - \partial_z$, and is holomorphic for $\operatorname{Re} z > 0$.

Set $\beta_z^0(x, y) = p_z(x, y)$ and for $\phi \in L^\infty(M)$ define the kernels

$$\beta_z^k(x, y) = \int_0^1 \int_M z p_{(1-v)z}(x, w)\, \beta_{zv}^{k-1}(w, y)\, \phi(w)\, dw\, dv, \qquad k = 1, 2, ...,$$

and denote by H_z^k the integral operator with kernel

$$G_z^k(x, y) = \int_0^1 z p_{(1-v)z}(x, y)\, \beta_{vz}^{k-2}(x, y)\, dv \qquad k \geqslant 2.$$

Clearly, when z is real β_z^k and H_z^k coincide with the corresponding real time objects defined in (16), (17).

For $0 \leqslant \theta_0 < \pi/2$ we define

$$S(\theta_0, T) = \{ z \in \mathbf{C} \colon |\arg z| \leqslant \theta_0, 0 < |z| \leqslant T \}$$

and for $\operatorname{Re} z > 0$ we set

$$t = |z|.$$

We shall assume throughout that $f \in L^2(M)$, $\int_M f = 0$.

Estimate (18) is a special case of the following theorem:

THEOREM 6. *Let $T > 0$ and $\theta_0 \in (0, \pi/2)$. There exist constants $\alpha = \alpha(\theta_0)$, $C = C(\theta_0, T)$, such that, $\forall z \in S(\theta_0, T)$, $k \geqslant 2$ and $\ell \geqslant 0$*

$$|\partial_z^\ell (H_z^k f, f)| \leqslant C^{k+\ell+1} \frac{\ell!}{k!}\, t^{k-2-n/2-\ell}\, \|\phi\|_\infty^{k-2}\, \|\Delta^{-(1/2)} f\|_2^2.$$

The strategy is the same as the one used in [M1]: prove the estimates for the case $\ell = 0$ and then use the Cauchy integral formula to obtain the bounds for higher order time derivatives.

The first step is to study the kernel G_z^k; in the next theorem we show that it is essentially a Green's kernel.

Below, we adopt the same notation as in [M1]. In particular, $d(x, y)$ denotes geodesic distance, and $\nabla_x^i \nabla_y^j f$ denote mixed covariant derivatives of f.

THEOREM 7. *Let $T>0$ and $\theta_0\in(0,\pi/2)$. There exist constants $\alpha=\alpha(\theta_0)$, $C=C(\theta_0,T)$, such that, $\forall z\in S(\theta_0,T)$ and $\forall x,y\in M$*

$$\text{(a)}\quad |\nabla_x^i\nabla_y^j G_z^k(x,y)| \leqslant C^{k+1}\,\|\phi\|_\infty^{k-2}\,\frac{1}{k!}\,t^{k-2-n/2}\,d(x,y)^{2-n-i-j}\,e^{-d^2(x,y)/\alpha t}$$

for $i,j=0,1$ and $k\geqslant 2$, with the exception $n=2$, $i=j=0$, in which case $d(x,y)^{2-n}$ is replaced by $1+|\log d(x,y)|$.

If in addition $\delta\in(0,1)$ then for a suitable $C(\theta_0,T,\delta)$

$$\text{(b)}\quad |\nabla_y\nabla_x G_z^k(x,y)-\nabla_y\nabla_x G_z^k(x',y)| \leqslant C^{k+1}\,\|\phi\|_\infty^{k-2}\,\frac{1}{k!}\,t^{k-2-n/2}\,d(x,x')^{\delta}\,d(x,y)^{-n-\delta}\,e^{-d^2(x,y)/\alpha t}$$

for $d(x,x')\leqslant d(x,y)/2$.

The proof of this theorem is based on the following Gaussian estimates on β_z^k, derived in [M1], Theorem 1:

THEOREM 8 [M1]. *For $z\in S(\theta_0,T)$, $x,y\in M$, $k,\ell\geqslant 0$, and $i,j=0,1$ we have*

$$|\nabla_y^j\nabla_x^i\,\partial_z^\ell\,\beta_z^k|\leqslant C^{k+\ell+1}\,\|\phi\|_\infty^k\,\frac{\ell!}{k!}\,t^{k-n/2-\ell-(i+j)/2}\,e^{-d^2(x,y)/\alpha t}.$$

If in addition $\delta\in(0,1)$, then

$$|\nabla_x^i\nabla_y^j\beta_z^k(x,y)-\nabla_x^i\nabla_y^j\beta_z^k(x',y)| \leqslant C^{k+1}\,\|\phi\|_\infty^k\,\frac{d(x,x')^\delta}{k!}\,t^{k-n/2-(i+j+\delta)/2}\,e^{-d^2(x,y)/\alpha t}$$

for $i,j=0,1$, and $d(x,x')\leqslant d(x,y)/2$.

Proof of Theorem 7. It is straightforward to check that for $d\neq 0$ and $\alpha>0$

$$\int_0^t s^{-\gamma}e^{-d^2/\alpha s}\,ds\leqslant W_\gamma(t)\,e^{-d^2/2\alpha t}$$

where

$$W_\gamma(t) = C(\alpha, \gamma) \begin{cases} d^{2-2\gamma} & \text{if} \quad \gamma > 1 \\ 1 + |\log(d^2/t)| & \text{if} \quad \gamma = 1 \\ t^{1-\gamma} & \text{if} \quad -\infty < \gamma < 1. \end{cases}$$

For the sake of simplicity we outline the proof of (a) only for the case $j = 0$, $i = 1$. The remaining cases are treated similarly.

From Leibniz's rule we can estimate

$$|\nabla_x G_z^k| \leqslant C \int_0^1 t \, |\nabla_x p_{(1-v)z}| \, |\beta_{vz}^{k-2}| \, dv + \int_0^1 t \, |p_{(1-v)z}| \, |\nabla_x \beta_{vz}^{k-2}| \, dv = I + II.$$

From Theorem 8, setting $d = d(x, y)$,

$$I \leqslant C \int_0^1 t(t(1-v))^{-(n+1)/2} \, (tv)^{k-2-n/2} \, e^{-d^2/\alpha t(1-v)} e^{-d^2/\alpha tv} \, dv$$

$$= \int_0^t (t-s)^{-(n+1)/2} \, s^{k-2-n/2} e^{-d^2/\alpha(t-s)} e^{-d^2/\alpha s} \, ds$$

$$\leqslant C t^{k-2} \int_0^t (t-s)^{-(n+1)/2} e^{-d^2/\alpha(t-s)} e^{-d^2/\alpha s} \, ds$$

$$\leqslant C t^{k-2} \left(\int_0^{t/2} + \int_{t/2}^t \right) \leqslant C t^{k-2-n/2} [t^{-1/2} W_{n/2}(t) + W_{(n+1)/2}(t)] \, e^{-d^2/\alpha t}.$$

Hence, if $n > 2$ (our constants C, α may vary from place to place)

$$I \leqslant C t^{k-2-n/2} [t^{-1/2} d^{2-n} + d^{1-n}] \, e^{-d^2/\alpha t} \leqslant C t^{k-2-n/2} d^{1-n} e^{-d^2/\alpha t}.$$

If instead $n = 2$

$$I \leqslant C t^{k-3} [t^{-1/2} (1 + |\log(d^2/t)|) + d^{1-n}] \, e^{-d^2/\alpha t} \leqslant C t^{k-3} d^{-1} d^{-d^2/\alpha t}.$$

The argument for II is similar.

To prove (b), we apply the Leibniz rule again

$$|\nabla_y \nabla_x G_z^k(x, y) - \nabla_y \nabla_x G_z^k(x', y)|$$

$$\leqslant \sum_{\substack{r' + s' = 1 \\ r + s = 1}} \int_0^1 t \, |\nabla_y^{r'} \nabla_x^r p_{(1-v)z}(x, y) - \nabla_y^{r'} \nabla_x^r p_{(1-v)z}(x', y)|$$

$$\times |\nabla_y^{s'} \nabla_x^s \beta_{vz}^{k-2}(x, y)| + t \, |\nabla_y^{r'} \nabla_x^r p_{(1-v)z}(x', y)|$$

$$\times |\nabla_y^{s'} \nabla_x^s \beta_{vz}^{k-2}(x, y) - \nabla_y^{s'} \nabla_x^s \beta_{vz}^{k-2}(x', y)| \, dv.$$

For simplicity we only consider one of these terms:

$$\int_0^1 t\,|p_{(1-v)z}(x', y)|\,|\nabla_y \nabla_x \beta_{vz}^{k-2}(x, y) - \nabla_y \nabla_x \beta_{vz}^{k-2}(x', y)|\,dv$$

$$\leqslant C \int_0^1 t(t(1-v))^{-n/2}\,(tv)^{k-3-(n+\delta)/2}\,d(x, x')^{\delta}$$

$$\times e^{-d^2(x', y)/\alpha t(1-v)} e^{-d^2(x, y)/\alpha tv}\,dv$$

$$\leqslant Cd(x, x')^{\delta} \int_0^t (t-s)^{-n/2}\,s^{k-3-(n+\delta)/2} e^{-d^2(x, y)/\alpha(t-s)} e^{-d^2(x, y)/\alpha s}\,ds,$$

since $d(x', y) \geqslant d(x, y) - d(x, x') \geqslant d(x, y)/2$. At this point one can proceed as in the proof of (a) and obtain the desired estimate. We leave it to the reader to fill in the details, and to complete the proof of (b). ∎

To prove Theorem 6 we proceed as follows. Denote by ψ a (weak) solution of $\Delta\psi = -f$ (i.e. $\psi(x) = \int_M g(x, y)\,f(y)\,dy$). Then, $\|\Delta^{-(1/2)} f\|_2 = \|\nabla\psi\|_2$. By applying (formally) Green's formula we obtain

$$(H_z^k f, f) = \int_M \int_M G_z^k(x, y)\,\mathrm{div}\,\nabla\psi(x)\,\mathrm{div}\,\nabla\psi(y)\,dx\,dy$$

$$= \int_M \int_M \nabla_x \nabla_y G_z^k \cdot \nabla_x \psi \cdot \nabla_y \psi\,dx\,dy.$$

From Theorem 7, we know that G_z^k is a Green-like kernel, in particular the best bound it satisfies seems to be $|\nabla_x \nabla_y G_z^k(x, y)| \leqslant Cd(x, y)^{-n}$, i.e. if the RHS above has any meaning, it must be in the sense of singular integrals. Hence, the task is to show that the bilinear singular form on the RHS, defined on the space $\mathscr{X}$ of C^∞ vector fields on M, is bounded on $L^2 \times L^2$. The result that we will need is the following.

THEOREM 9. *Let $K(x, y)$ be complex-valued kernel, satisfying $K(x, y) = K(y, x)$ and*

$$|\nabla_y^j \nabla_x^i K(x, y)| \leqslant C^* d(x, y)^{2-n-i-j}, \qquad i, j = 0, 1$$

$$|\nabla_y \nabla_x K(x, y) - \nabla_y \nabla_x K(x', y)| \leqslant C^* d(x, x')^{\delta}\,d(x, y)^{-n-\delta}, \qquad \delta \in (0, 1)$$

whenever $x \neq y$ and $d(x, x') \leqslant d(x, y)/2$, with the exception $n = 2$, $i = j = 0$, in which case we require $|K(x, y)| \leqslant C^*(1 + |\log d(x, y)|)$. Here $C^* = C^*(\delta) > 0$.

Then the bilinear form

$$T(X, Y) = \int_M \int_M K(x, y) \operatorname{div}_x X \operatorname{div}_y Y \, dx \, dy$$

is well-defined and bounded in $L^2(\mathscr{X}) \times L^2(\mathscr{X})$, *with*

$$|T(X, Y)| \leqslant C_\delta C^* \|X\|_2 \|Y\|_2.$$

Recall that $\|X\|_2^2 = \int_M X \cdot X \, dx$, with $X \cdot Y = g(X, Y)$.

The proof of this theorem is a more or less straightforward application of the celebrated T1-Theorem of G. David and J. L. Journé [D-J], which we shall now recall. The version below is taken from Christ-Journé [C-J].

Let $K: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ be a kernel with the following properties:

$$|K(x, y)| \leqslant c \, |x - y|^{-n} \tag{i}$$

$$|K(x, y) - K(x', y)| \leqslant c \, |x - x'|^\delta \, |x - y|^{-n-\delta} \tag{ii}$$

$$|K(y, x) - K(y, x')| \leqslant c \, |x - x'|^\delta \, |x - y|^{-n-\delta} \tag{iii}$$

for some $\delta > 0$, all $x \neq y$ and $|x - x'| \leqslant |x - y|/2$.

The bilinear singular form associated to K is defined as

$$T(g, f) = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} K(x, y) \, g(x) \, f(y) \, dx \, dy$$

for $f, g \in C_0^\infty(\mathbf{R}^n)$ with disjoint supports. This T can be extended (see [D-J]) to $C_{00}^\infty \times (C^\infty \cap L^\infty)$ or $(C^\infty \cap L^\infty) \times C_{00}^\infty$, where C_{00}^∞ denotes the subspace of C_0^∞ of functions with vanishing integral. We denote by $\mathbf{1}$ the function identically equal to one and by $T_1 \mathbf{1}$ the element of $[C_{00}^\infty]'$ defined by $\langle g, T_1 \mathbf{1} \rangle = T(g, \mathbf{1})$. Similarly, define $T_2 \mathbf{1}$ by duality.

The form T is said to have the *weak boundedness property*, if for all $f, g \in C_0^\infty(\mathbf{R}^n)$ whose supports have diameter at most $4t$

$$|T(g, f)| \leqslant ct^n(\|g\|_\infty + t \, \|\nabla g\|_\infty)(\|f\|_\infty + t \, \|\nabla f\|_\infty) \tag{iv}$$

and it is *bounded* if

$$|T(g, f)| \leqslant c \, \|f\|_2 \, \|g\|_2 \tag{v}$$

The best constant c in (i)–(iii) is denoted by $|T|_\delta$ and the best constants in (iv), (v) are denoted by $|T|_W$, $\|T\|_{2,2}$, respectively.

T1-THEOREM [D-J]. *The bilinear singular form T defined above is bounded if and only if $T_1\mathbf{1}$, $T_2\mathbf{1}\in BMO$ and T has the weak boundedness property. In this case we have*

$$\|T\|_{2,2}\leqslant c(\|T_1\mathbf{1}\|_{\mathrm{BMO}}+\|T_2\mathbf{1}\|_{\mathrm{BMO}}+|T|_W)+c_\delta\,|T|_\delta$$

with c, c_δ depending only on the dimension and δ.

BMO is the space of functions with bounded mean oscillation of John and Niremberg (see [J-N] for definitions and properties).

Proof of Theorem 9. By a standard localization argument using partition of unity, one is reduced to show boundedness of a bilinear singular integral of type

$$\tilde{T}_{k\ell}(a,b):=\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\tilde{K}(\xi,\eta)\,\frac{\partial a}{\partial\xi_k}\,\frac{\partial b}{\partial\eta_\ell}\,d\xi\,d\eta,\qquad 1\leqslant k,\ell\leqslant n$$

where a, b are C^∞ functions compactly supported in the unit ball B, and where $\tilde{K}$ is a kernel satisfying (i)–(iii), compactly supported in $B\times B$ (see [M2] for more details).

The bound $|\tilde{T}_{k\ell}(a,b)|\leqslant CC^*\,\|a\|_2\,\|b\|_2$, with C independent from a, b, follows from the T1-Theorem, provided $\tilde{T}_{k\ell}$ satisfies the weak boundedness property (obviously $(\tilde{T}_{k\ell})_1\,\mathbf{1}=(\tilde{T}_{k\ell})_2\,\mathbf{1}=0\in\mathrm{BMO}$). But this follows quickly, since if $a, b\in C^\infty(\mathbf{R}^n)$ are supported in cubes Q_t, Q'_t of diameter at most $4t$, centered, say, at ξ_0, η_0, then for $|\xi_0-\eta_0|\geqslant 2t$

$$\begin{aligned}|\tilde{T}_{k\ell}(a,b)|&\leqslant CC^*\,\|\nabla a\|_\infty\,\|\nabla b\|_\infty\int_{Q_t}\int_{Q'_t}|\xi-\eta|^{2-n}\,d\xi\,d\eta\\&\leqslant CC^*\,\|\nabla a\|_\infty\,\|\nabla b\|_\infty\,t^{n+2},\end{aligned}$$

whereas for $|\xi_0-\eta_0|<2t$

$$\begin{aligned}|\tilde{T}_{k,\ell}(a,b)|&\leqslant CC^*t^{n+2}\,\|\nabla a\|_\infty\,\|\nabla b\|_\infty\int_Q\int_Q\left|\xi-\eta+\frac{\xi_0-\eta_0}{t}\right|^{2-n}d\xi\,d\eta\\&\leqslant CC^*t^{n+2}\,\|\nabla a\|_\infty\,\|\nabla b\|_\infty.\quad\blacksquare\end{aligned}$$

Proof of Theorem 6. From Theorem 7 we know that the kernel $G_z^k(x,y)$ satisfies the hypothesis of Theorem 9. Hence, by applying Theorem 9 with, say, $\delta=1/2$, we obtain the case $\ell=0$. To prove the estimate for arbitrary ℓ, let $z\in S(\theta_0,T)\subset S(\theta_1,2T)$ some $\theta_1\in(\theta_0,\pi/2)$. The circle γ centered at z

and with radius $t \sin(\theta_1 - \theta_0)$ is contained in $S(\theta_1, 2T)$. By the Cauchy's integral formula and the estimate for $\ell = 0$ in the sector $S(\theta_1, 2T)$

$$
\begin{aligned}
|\partial_z^\ell (H_z^k f, f)| &\leqslant \frac{\ell\,!}{(t \sin(\theta_1 - \theta_0))^\ell} \max_{w \in \gamma} |(H_w^k f, f)| \\
&\leqslant C^{k+\ell+1} \frac{\ell\,!}{k!} t^{k-2-n/2-\ell} \, \|\phi\|_\infty^{k-2} \, \|\Delta^{-(1/2)} f\|_2^2. \quad \blacksquare
\end{aligned}
$$

5. PROOF OF THEOREM 1

We first recall a few facts concerning the Möbius group and the "center of mass argument". We follow closely [C-Y].

Identify S^2 with the z-plane via the stereographic projection. The conformal transformations of S^2 are identified with the Möbius transformations

$$
\tau = \frac{az + b}{cz + d}, \qquad ad - bc = 1, \qquad a, b, c, d \in \mathbf{C},
$$

which form a 6-dimensional Lie group, denoted by $\mathscr{M}$.

Given $P \in S^2$ and $\lambda > 0$ denote by $\tau_{P,\lambda}$ the Möbius transformation that in stereographic coordinates with P at infinity is given by

$$
\tau_{P,\lambda} = \lambda z
$$

and define

$$
\mathscr{U} = \{\tau_{P,\lambda} \colon P \in S^2, \lambda \geqslant 1\}.
$$

As in [C-Y], we identify $\mathscr{U}$ with the open unit ball $B^3 \subset \mathbf{R}^3$ via

$$
(Q, \lambda) \in S^2 \times [1, \infty) \leftrightarrow \frac{\lambda - 1}{\lambda} Q \in B^3.
$$

In the notation of Section 1, given $w \in C^\infty(S^2)$ define $w_\tau = (w \circ \tau)\, |\tau'|$, where $|\tau'|$ is the conformal factor of the metric due to the change of coordinates $z \to \tau(z)$. In stereographic coordinates the volume density of the round metric of S^2 is $4(1 + |z|^2)^{-2}\, |dz|$, so that we find, in stereographic coordinates,

$$
|\tau'(z)| = \left(\frac{1 + |z|^2}{|az + b|^2 + |cz + d|^2} \right)^2
$$

The *center of mass* of w is given by the vector of $\mathbf{R}^3$

$$\mathbf{cm}(w) = \frac{1}{4\pi}\int_{S^2} \mathbf{x} w \, dx$$

where $\mathbf{x} = (x_1, x_2, x_3)$. Since the spherical harmonics of degree 1 are linear combinations of the coordinates, we have $\mathbf{cm}(w) = 0 \Leftrightarrow w \in \mathscr{H}_1^{\perp}$.

In [C-Y], it is proved that for all $w \in W^{1,2}$ there exists $\tau \in \mathscr{U}$ such that $\mathbf{cm}(w_\tau) = 0$. The proof makes use of properties of the map T given by

$$T(w, \tau) = \int_{S^2} (\mathbf{x} \circ \tau) \, w \, dx = \int_{S^2} \mathbf{x} w_{\tau^{-1}} \, dx.$$

Observe that $T(w, \tau) = 0$ if and only if $\mathbf{cm}(w_{\tau^{-1}}) = 0$. Moreover, in the proof of Proposition 2.2 of [C-Y], Chang and yang showed (essentially) that the differential $\partial_\tau T|_{\tau = \text{id.}}$ is an invertible linear map from $\mathbf{R}^3$ to $\mathbf{R}^3$. Recall that $\mathscr{U} \approx B^3$ and that the identity corresponds to $0 \in \mathbf{R}^3$.

Now observe that we can view T as a C^∞ map defined in $L^\infty \times \mathscr{U}$. Since $T(1, \text{id.}) = 0$, from the implicit function theorem we deduce that $T(w, \tau) = 0$ defines implicitly a C^∞ function $\tau = \tau(w)$, defined in a ball $B(1, \varepsilon) = \{\|w - 1\|_\infty < \varepsilon\} \subset L^\infty(S^2)$, for some $\varepsilon > 0$. In this ball we have $\mathbf{cm}(w_{\tau(w)^{-1}}) = 0$.

Proof of Theorem 1. Let $0 < w \in C^\infty(S^2)$, $\int_{S^2} w = 4\pi$. If $\mathbf{cm}(w) = 0$, from Corollary 1 and Theorem 2 we deduce that for $b > a > 0$ there is a $C_0 = C_0(a, b) > 0$ such that

$$Tr(e^{t\Delta_w}) - Tr(e^{t\Delta}) \geqslant 0, \qquad \forall t \in [a, b],$$

provided $\|w - 1\|_\infty < C_0$. In this case equality holds for some $t \in [a, b]$ only if $w \equiv 1$.

Let now w be arbitrary and use the C^∞ function τ constructed above. Since τ' is continuous at $w = 1$, so is $|\tau'|$, hence there exists $\varepsilon_0 = \varepsilon_0(a, b) > 0$ such that

$$\|w - 1\|_\infty < \varepsilon_0 \Rightarrow \|w_{\tau(w)^{-1}} - 1\|_\infty = \|w \, |\tau(w)'|^{-1} - 1\|_\infty < C_0$$

Let $M[w] = \inf_{\tau \in \mathscr{M}} \|w_\tau - 1\|_\infty < \varepsilon_0$. By continuity, $M[w] = \|w_{\tau_0} - 1\|_\infty$, for some $\tau_0 \in \mathscr{M}$. Since the trace is invariant under conformal transformations we can write

$$Tr(e^{t\Delta_w}) - Tr(e^{t\Delta}) = Tr(e^{t\Delta_\rho}) - Tr(e^{t\Delta}), \qquad \text{with} \quad \rho = (w_{\tau_0})_{\tau(w_{\tau_0})^{-1}}.$$

However, the RHS of this identity is $\geqslant 0$, for $t \in [a, b]$, since by the properties of $\tau(w)$ and the choice of ε_0 the function ρ has center of mass 0 and $\|\rho - 1\|_\infty < C_0$. Equality can only occur for $\rho = (w_{\tau_0})_{\tau(w_{\tau_0})^{-1}} \equiv 1$. After a change of variable, this condition is the same as $w_{\tau_0} = |\tau(w_{\tau_0})'|$, which, after another change of variable, is the same as $w = |(\tau(w_{\tau_0}) \circ \tau_0^{-1})'|$. But this means $M[w] = 0$. ▮

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